

Feedback stabilisation of switched systems via iterative approximate eigenvector assignment*

Hernan Haimovich[†] and Julio H. Braslavsky[‡]

Abstract

This paper presents and implements an iterative feedback design algorithm for stabilisation of discrete-time switched systems under arbitrary switching regimes. The algorithm seeks state feedback gains so that the closed-loop switching system admits a common quadratic Lyapunov function (CQLF) and hence is uniformly globally exponentially stable. Although the feedback design problem considered can be solved directly via linear matrix inequalities (LMIs), direct application of LMIs for feedback design does not provide information on closed-loop system structure. In contrast, the feedback matrices computed by the proposed algorithm assign closed-loop structure approximating that required to satisfy Lie-algebraic conditions that guarantee existence of a CQLF. The main contribution of the paper is to provide, for single-input systems, a numerical implementation of the algorithm based on iterative approximate common eigenvector assignment, and to establish cases where such algorithm is guaranteed to succeed. We include pseudocode and a few numerical examples to illustrate advantages and limitations of the proposed technique.

1 Introduction

We consider the discrete-time switching system (DTSS)

$$x_{k+1} = A_{i(k)}x_k + B_{i(k)}u_k, \quad (1)$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, defined by a switching function

$$i(k) \in \underline{n} := \{1, 2, \dots, n\}, \quad \text{for all } k,$$

and a set of controllable subsystem pairs $\{(A_i, B_i) : i \in \underline{n}\}$, where the input matrices B_i , $i \in \underline{n}$, have full column rank.

We address feedback control design of the form

$$u_k = K_{i(k)}x_k, \quad (2)$$

*Version from September 13, 2010. Extended version of that submitted to CDC 2010.

[†]H. Haimovich is with CONICET and the Laboratorio de Sistemas Dinámicos y Procesamiento de Información, Departamento de Control, Escuela de Ingeniería Electrónica, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Riobamba 245bis, 2000 Rosario, Argentina, haimo@fceia.unr.edu.ar

[‡]J.H. Braslavsky is with the ARC Centre for Complex Dynamic Systems and Control, The University of Newcastle, Callaghan NSW 2308, Australia jhb@ieee.org

(which assumes that at every time instant k the “active” subsystem given by $i(k)$ is known) so that the resulting closed-loop system

$$x_{k+1} = A_{i(k)}^{\text{cl}} x_k, \quad \text{where} \quad (3)$$

$$A_i^{\text{cl}} = A_i + B_i K_i, \quad \text{for } i \in \underline{n}, \quad (4)$$

be exponentially stable under arbitrary switching.

It is well-known that ensuring that the closed-loop matrices A_i^{cl} are stable for each $i \in \underline{n}$ is necessary but not sufficient to ensure the stability of the DTSS (3)–(4) under arbitrary switching [1]. A necessary and sufficient condition for uniform exponential stability under arbitrary switching is the existence of a common Lyapunov function for each of the component subsystems in (3)–(4) [2]. Such Lyapunov functions, however, will in general have complex level sets, which makes their numerical search difficult [3].

The search for common *quadratic* Lyapunov functions (CQLF), although restrictive, is appealing, since these functions play an important role in the stabilisation of linear time-invariant systems such as the component subsystems in (1). The design of feedback matrices K_i in (2) so that the DTSS (3)–(4) admits a CQLF may be posed as follows.

Problem 1. *Given the matrices $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$ for $i \in \underline{n}$, design feedback matrices $K_i \in \mathbb{R}^{m \times n}$ such that the DTSS closed-loop system (3)–(4) admits a CQLF.*

Quadratic Lyapunov functions are amenable to efficient numerical computation using linear matrix inequalities (LMIs). For example, Problem 1 can be solved by finding $X = X^T > 0$ and N_i to satisfy the LMIs

$$\begin{bmatrix} X & (A_i X + B_i N_i)^T \\ A_i X + B_i N_i & X \end{bmatrix} > 0, \quad i \in \underline{n}, \quad (5)$$

where the required feedback matrices are given by $K_i = N_i X^{-1}$ and the CQLF is $V(x) = x^T X^{-1} x$ (see for example [4], [5]). An advantage of this approach is that the feasibility of the LMIs (5) is necessary and sufficient for the DTSS considered to admit a CQLF. However, blind application of such control design strategy gives no insight on the structure of the closed-loop DTSS. Thus, as pointed out in [6], these LMI methods “lack transparency and interpretability that was a feature of classical techniques” and hence “a pressing need remains for analytic tools to support the design of stable switched systems” (also see [3], [7] for similar comments).

As an alternative to the LMI approaches, the authors in [6] propose a pole-placement technique for single-input single-output continuous-time switching systems. The strategy in [6] seeks to guarantee closed-loop uniform global exponential stability under arbitrary switching by designing controllers that achieve a closed-loop common eigenvector structure. By constraining such eigenstructure and the class of controllers allowed, the strategy in [6] simplifies the design process, providing analytically transparent solutions in a restricted but practically important class of systems.

The present paper presents another strategy that seeks to “activate” analytic tools into a feedback design methodology (much in the spirit of [8]) to solve Problem 1. Our strategy follows from the previous paper [9], which introduced an iterative algorithm to seek feedback gains that make the set of closed-loop subsystems (3)–(4) satisfy Lie algebraic conditions that guarantee the existence of a CQLF [10]. While such Lie-algebraic conditions are restrictive, since they are not necessary for the existence of a

CQLF, we believe they offer an insightful way to understand and exploit fundamental system structure in feedback design for DTSS.

The theoretical results in [9] brought forward the following important consequences: (i) if the proposed Lie-algebraic feedback design problem is feasible, its solution can be found in an iterative manner (similarly to the way solvability of Lie-algebras can be checked for autonomous switched systems) by seeking feedback gains that assign a single common stable eigenvector at each iteration step; and (ii) in seeking such common eigenvector, if at any iteration step more than one vector can be assigned by feedback, it is irrelevant which one is chosen by the procedure. These observations provide motivation to the development of numerical implementations, which were not discussed in [9].

The present paper focuses on the numerical implementation aspects of the iterative algorithm introduced in [9]. A key question in the proposed approach is the lack of robustness of the Lie-algebraic conditions that are sought to satisfy by feedback design. Indeed, it is well-known that these conditions are destroyed by arbitrarily small perturbations to the individual matrices [10, §2.2.4], and thus, there are a priori no guarantees that the algorithm in [9] can find any solution at all in a (necessarily approximate) numerical implementation.

However, suppose that for a given set of systems to be stabilised there exists, in a neighbourhood of the original, a feasible (a priori unknown) set of systems, which is stabilisable and such that the resulting closed-loop systems satisfy the Lie-algebraic conditions that guarantee the existence of a CQLF. Then, by continuity of such CQLF, and if the neighbourhood is sufficiently small, there will also exist a CQLF for the original set of systems, despite the fact that it may be impossible to make them satisfy the Lie-algebraic conditions.

The main contribution of the present paper is to, based on the above argument, derive and mathematically justify a specific numerical implementation, for single-input systems, of the design algorithm proposed in [9] that will succeed not only in cases where the Lie-algebraic conditions are satisfied but also in approximate cases. The proposed numerical implementation is based on the solution of an optimisation problem that seeks feedback matrices that will achieve closed-loop systems with an *approximate* common eigenvector. The existence of a CQLF for the corresponding closed-loop DTSS may be readily checked a posteriori with a set of *informed* LMIs. This step is necessary, since it is in general a priori unknown if the “exact” problem is feasible in some neighbourhood of the original system data. The algorithm has been implemented in MATLAB®, and numerical examples to illustrate its application and discuss its advantages and limitations are provided.

The rest of the paper is organised as follows. The proposed algorithm and its core, a procedure for approximate common eigenvector assignment (Procedure CEA), are presented and explained in Section 2. The main theoretical results, justifying the numerical implementation of the proposed algorithm for single-input systems, appear in Section 3. Section 4 presents details about the MATLAB® implementation of the algorithm and some illustrative numerical examples, and Section 5 gives the paper conclusions. The main proofs (Theorems 2 and 3) are given in the Appendix.

Notation \mathbb{R} and \mathbb{C} denote the real and complex numbers. $\|\cdot\|$ denotes Euclidean norm or corresponding induced matrix norm. M^* denotes the conjugate transpose of M . $\rho(\cdot)$ denotes spectral radius and $(M)_{:,k}$ is the k -th column of M . If $M \in \mathbb{C}^{n \times m}$, $\text{Im } M$ denotes $\{x \in \mathbb{C}^n : x = My, y \in \mathbb{C}^m\}$. The Euclidean distance between a vector $v \in \mathbb{C}^n$ and a set $\mathcal{V} \subset \mathbb{C}^n$ is denoted $d(v, \mathcal{V})$. j denotes $\sqrt{-1}$.

Contents

1	Introduction	1
2	Feedback Control Design	4
2.1	The algorithm	4
2.2	Approximate common eigenvector assignment	6
3	Main Results	8
4	Examples	11
4.1	Randomly created DTSS	11
4.2	DTSS with no CQLF	12
4.3	CQLF exists but Algorithm 1 fails	12
5	Conclusions	12
A	Appendix	13
A.1	Proof of Theorem 2	13
A.1.1	Proof of Theorem 2 i)	14
A.1.2	Proof of Theorem 2 ii)	16
A.2	Proof of Theorem 3	17

2 Feedback Control Design

A sufficient condition for the closed-loop DTSS (3)–(4) to admit a CQLF is given by the following result, which is a minor modification of [11, Theorem 6.18].

Lemma 1. *If $\rho(A_i^{\text{cl}}) < 1$ for $i \in \underline{n}$, and the Lie algebra generated by $\{A_i^{\text{cl}} : i \in \underline{n}\}$ is solvable, then (3)–(4) admits a CQLF.*

In matrix terms, the fact that the Lie algebra generated by $\{A_i^{\text{cl}} : i \in \underline{n}\}$ be solvable is equivalent to the existence of a single invertible matrix $T \in \mathbb{C}^{n \times n}$ such that $T^{-1}A_i^{\text{cl}}T$ is upper triangular for $i \in \underline{n}$ (even if A_i^{cl} have real entries, those of T may be complex [12]).

2.1 The algorithm

In [9], we established that given A_i and B_i , there exist feedback matrices K_i that cause the Lie algebra generated by A_i^{cl} to be solvable if and only if such feedback matrices can be computed by an algorithm that performs iterative common eigenvector assignment by feedback. A matrix version of such algorithm is given in pseudocode as Algorithm 1.

At every iteration, Algorithm 1 executes Procedure CEA [see (6)]. This procedure attempts to find feedback matrices F_i^ℓ and a vector v_1^ℓ so that $(A_i^\ell + B_i^\ell F_i^\ell)v_1^\ell = \lambda_i^\ell v_1^\ell$ and $|\lambda_i^\ell| < 1$ for $i \in \underline{n}$, i.e. so that v_1^ℓ becomes a common eigenvector of a set of closed-loop matrices, with corresponding stable eigenvalues. The parameters ϵ_c and ϵ_d given as arguments to Procedure CEA are required for numerical reasons, and will be explained in Section 2.2.

Algorithm 1: Approximate triangularisation by feedback**Data:** $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times 1}$ for $i \in \underline{n}$, $\epsilon_c > 0$, $\epsilon_d > 0$ **Output:** U , K_i for $i \in \underline{n}$ **begin** Initialisation
$$\begin{aligned} & A_i^1 := A_i, B_i^1 := B_i, K_i^0 := 0, U_1 := I; \\ & U := [] \text{ (empty)}, \ell \leftarrow 0; \end{aligned}$$
repeat

$$\ell \leftarrow \ell + 1, \quad n_r \leftarrow n - \ell + 1$$

$$[v_1^\ell, \{F_i^\ell\}_{i=1}^n] \leftarrow \text{CEA}(\{A_i^\ell, B_i^\ell\}_{i=1}^n, \epsilon_c, \epsilon_d) \quad (6)$$

Define

$$A_i^{\ell, \text{CL}} := A_i^\ell + B_i^\ell F_i^\ell, \quad (7)$$

$$(U)_{:, \ell} \leftarrow \left(\prod_{r=\ell}^{r=1} U_r \right) v_1^\ell = U_1 U_2 \cdots U_\ell v_1^\ell,$$

$$K_i^\ell \leftarrow K_i^{\ell-1} + F_i^\ell \left(\prod_{r=1}^{\ell} U_r^* \right) \quad (8)$$

if $\ell < n$ **then**Construct unitary matrix with first column v_1^ℓ :

$$[v_1^\ell | v_2^\ell | \cdots | v_{n_r}^\ell] \in \mathbb{C}^{n_r \times n_r}. \quad (9)$$

Assign

$$U_{\ell+1} \leftarrow [v_2^\ell | \cdots | v_{n_r}^\ell], \quad (10)$$

$$A_i^{\ell+1} \leftarrow U_{\ell+1}^* A_i^{\ell, \text{CL}} U_{\ell+1}, \quad (11)$$

$$B_i^{\ell+1} \leftarrow U_{\ell+1}^* B_i^\ell, \quad (12)$$

until $\ell = n$; $K_i \leftarrow K_i^n$;

Remark 1. It is straightforward to check that if Algorithm 1 terminates successfully and at every iteration ($\ell = 1, \dots, n$) Procedure CEA is able to find the vector v_1^ℓ and feedback matrices F_i^ℓ such that $(A_i^\ell + B_i^\ell F_i^\ell)v_1^\ell = \lambda_i^\ell v_1^\ell$ for $i \in \underline{n}$, then the matrices A_i^{CL} given by (4) with K_i as computed by the algorithm are such that $U^* A_i^{\text{CL}} U$ are upper triangular and λ_i^ℓ is the ℓ -th main-diagonal entry of $U^* A_i^{\text{CL}} U$.

Note that a slight modification of Algorithm 1 is necessary to ensure that real feedback matrices K_i are computed. We do not explain such modification here due to space limitations, and because it does not add any essential information to our main results. The implemented computational version of the algorithm, employed in Section 4, does indeed ensure such condition.

Our result in [9] established that a vector v_1^ℓ and feedback matrices F_i^ℓ such that $(A_i^\ell + B_i^\ell F_i^\ell)v_1^\ell = \lambda_i^\ell v_1^\ell$

with $|\lambda_i^\ell| < 1$, for $i \in \underline{n}$, will exist at every iteration of Algorithm 1 if and only if there exist K_i such that A_i^{cl} as in (4) generate a solvable Lie algebra and satisfy $\rho(A_i^{\text{cl}}) < 1$. Such result was of a theoretical nature, since in a numerical implementation determining whether a vector is exactly an eigenvector or is close to being so is an extremely difficult task [13].

2.2 Approximate common eigenvector assignment

In this section, we provide a numerical implementation of Procedure CEA executed by Algorithm 1 and establish some of its properties. We give Procedure CEA in pseudocode first, and next define and explain each of its parts.

Procedure CEA

Input: $A_i^\ell \in \mathbb{C}^{n_r \times n_r}$, $B_i^\ell \in \mathbb{C}^{n_r \times 1}$, for $i \in \underline{n}$, ϵ_c , ϵ_d

Output: v_1^ℓ , F_i^ℓ for $i \in \underline{n}$

if $n_r = 1$ **then**

Select $A_i^{\ell, \text{cl}}$ such that $|A_i^{\ell, \text{cl}}| \leq 1 - \epsilon_c$;

$v_1^\ell \leftarrow 1$, $F_i^\ell \leftarrow -(B_i^\ell)^{-1} A_i^\ell + A_i^{\ell, \text{cl}}$;

else

if $\mathcal{S}(\epsilon_c, \epsilon_d) = \emptyset$ **then**

Stop: unsuccessful termination.

else

$v_1^\ell \leftarrow \operatorname{argmin}_{v \in \mathcal{S}(\epsilon_c, \epsilon_d)} J(v)$;

$F_i^\ell \leftarrow M_i(v_1^\ell)$;

Note that n_r is the state dimension, which decreases by 1 at every iteration of Algorithm 1. Hence, the case $n_r = 1$ in Procedure CEA corresponds to the trivial case of a one-dimensional single-input system. If every subsystem is controllable, then $0 \neq B_i^\ell \in \mathbb{C}$ and hence $A_i^{\ell, \text{cl}}$ can be arbitrarily chosen.

For the case $n_r > 1$, Procedure CEA utilises the matrices

$$E_i(v) := (vv^* - I)A_i^\ell, \quad (13)$$

$$H_i(v) := (vv^* - I)B_i^\ell, \quad (14)$$

$$M_i(v) := -(H_i(v)^* H_i(v))^{-1} H_i(v)^* E_i(v), \quad (15)$$

$$A_i^{\ell, \text{cl}}(v) := A_i^\ell + B_i^\ell M_i(v), \quad (16)$$

the sets

$$\mathcal{S}_1 := \{v \in \mathbb{C}^{n_r} : \|v\| = 1\} \quad (17)$$

$$\mathcal{S}_2(\epsilon_c) := \bigcap_{i=1}^N \{v \in \mathbb{C}^{n_r} : \|A_i^{\ell, \text{cl}}(v)v\| \leq 1 - \epsilon_c\}, \quad (18)$$

$$\mathcal{S}_3(\epsilon_d) := \bigcap_{i=1}^N \{v \in \mathbb{C}^{n_r} : d(v, \operatorname{Im} B_i^\ell) \geq \epsilon_d\}, \quad (19)$$

$$\mathcal{S}(\epsilon_c, \epsilon_d) := \mathcal{S}_1 \cap \mathcal{S}_2(\epsilon_c) \cap \mathcal{S}_3(\epsilon_d), \quad (20)$$

and the cost function

$$J(v) := \sum_{i=1}^N \left\| [E_i(v) + H_i(v)M_i(v)]v \right\|^2. \quad (21)$$

If $n_r > 1$, Procedure CEA checks whether the set $\mathcal{S}(\epsilon_c, \epsilon_d)$ given by (20) is empty. Comments on the case $\mathcal{S}(\epsilon_c, \epsilon_d) = \emptyset$ are given later in Remark 2. If $\mathcal{S}(\epsilon_c, \epsilon_d) \neq \emptyset$, Procedure CEA searches for a vector v_1^ℓ that minimises $J(v)$ as given by (21), subject to the constraint $v \in \mathcal{S}(\epsilon_c, \epsilon_d)$. The constraint set $\mathcal{S}(\epsilon_c, \epsilon_d)$ in (20) is the intersection of three sets: \mathcal{S}_1 in (17), which constrains the search to unit vectors; $\mathcal{S}_2(\epsilon_c)$ in (18), which imposes the stability constraint $|\lambda_i^\ell| < 1$, as we will shortly demonstrate; and $\mathcal{S}_3(\epsilon_d)$ in (19), which is included for technical reasons discussed next and justified in the next section.

Note that if $\epsilon_d > 0$ and $v \in \mathcal{S}_3(\epsilon_d)$, then (19) implies that $v \notin \text{Im } B_i^\ell$, for $i \in \underline{n}$. It follows that $H_i(v)$ has the same (column) rank as B_i^ℓ [see (14)] and, in this case, $M_i(v)$ in (15) is well-defined if B_i^ℓ has full column rank (i.e. columns).

Thus, Procedure CEA requires B_i^ℓ to have full column rank. At the first iteration of Algorithm 1, we have $B_i^1 = B_i$ and hence B_i^1 has full column rank by assumption. At subsequent iterations of the algorithm, such condition is ensured by the following result.

Lemma 2. *Let B_i^ℓ have full column rank and let v_1^ℓ be computed by Procedure CEA. Then, $B_i^{\ell+1}$ has full column rank.*

Proof. Since v_1^ℓ is effectively computed by Procedure CEA and since $\mathcal{S}(\epsilon_c, \epsilon_d)$ is topologically closed, then $v_1^\ell \in \mathcal{S}(\epsilon_c, \epsilon_d)$ and hence $v_1^\ell \notin \text{Im } B_i^\ell$. The result then follows straightforwardly from the fact that the columns of (9) form a basis, (10) and (12). \square

We are now ready to show in what sense the minimisation of $J(v)$ is related to the assignment of a common eigenvector.

Lemma 3. *Let $\epsilon_c > 0$, $\epsilon_d > 0$ and $v \in \mathcal{S}(\epsilon_c, \epsilon_d)$. Then,*

- i) $J(v) \geq 0$,
- ii) $J(v) = 0$ if and only if $A_i^{\ell, \text{cl}}(v)v = \lambda_i^\ell v$ with $|\lambda_i^\ell| \leq 1 - \epsilon_c$.
- iii) *There exists $G_i \in \mathbb{C}^{m \times n_r}$ such that*

$$(A_i^\ell + B_i^\ell G_i)v = \lambda_i^\ell v, \quad (22)$$

if and only if $A_i^{\ell, \text{cl}}(v)v = \lambda_i^\ell v$.

Proof. i) Straightforward from (21) and since J is well defined on $\mathcal{S}(\epsilon_c, \epsilon_d)$.

ii) (\Rightarrow) $J(v) = 0$ implies that

$$\begin{aligned} 0 &= \left\| [E_i(v) + H_i(v)M_i(v)]v \right\| \\ &= \left\| (v v^* - I)[A_i^\ell + B_i^\ell M_i(v)]v \right\|, \end{aligned} \quad (23)$$

for $i \in \underline{n}$, where we have used (13), (14) and (21). In turn, (23) implies that

$$[A_i^\ell + B_i^\ell M_i(v)]v = A_i^{\ell, \text{cl}}(v)v = \lambda_i^\ell v, \quad (24)$$

where we have used (16). From (24), $|\lambda_i^\ell| = \|A_i^{\ell, \text{cl}}(v)v\| \leq 1 - \epsilon_c < 1$ since $v \in \mathcal{S}(\epsilon_c, \epsilon_d)$.

(\Leftarrow) Left-multiply (24) by $(vv^* - I)$ to obtain (23), which implies $J(v) = 0$.

iii) (\Rightarrow) Left-multiplying (22) by $(vv^* - I)$ yields $[E_i(v) + H_i(v)G_i]v = 0$. Let $u_i := G_i v$. Then, $E_i(v)v = -H_i(v)u_i$. Since $H_i(v)$ has full column rank, then $u_i = M_i(v)v = G_i v$ and hence by (22) $A_i^{\ell, \text{cl}}(v)v = [A_i^\ell + B_i^\ell M_i(v)]v = \lambda_i^\ell v$.

(\Leftarrow) Just take $G_i = M_i(v)$. □

Lemma 3 shows that, for $v \in \mathcal{S}(\epsilon_c, \epsilon_d)$, $J(v) = 0$ if and only if v can be assigned by feedback as a common eigenvector with corresponding stable eigenvalues, and $J(v) > 0$ otherwise. Therefore, the search for a vector that minimises $J(v)$ can be interpreted as the search for a vector that is closest to an assignable common eigenvector.

Remark 2. If $\mathcal{S}(\epsilon_c, \epsilon_d) = \emptyset$, then no feasible vector exists for the optimisation performed by Procedure CEA. Therefore, no vector can be computed, and Procedure CEA and hence Algorithm 1 will terminate unsuccessfully. In the next section, we show that there exist $\epsilon_c, \epsilon_d > 0$ so that $\mathcal{S}(\epsilon_c, \epsilon_d) \neq \emptyset$ in the cases of interest.

In the next section, we establish conditions under which the vector and feedback matrices computed by Procedure CEA cause Algorithm 1 to yield feedback matrices so that the closed-loop DTSS admits a CQLF.

3 Main Results

We now derive several results that justify the use of Algorithm 1 for feedback control design of single-input systems. Recall that the constraint set $\mathcal{S}(\epsilon_c, \epsilon_d)$ in the optimisation solved in Procedure CEA forces the search to be performed over vectors v that satisfy $v \notin \text{Im } B_i^\ell$. The following result justifies this constraint for single-input DTSS with controllable subsystems.

Lemma 4. Let $n_r > 1$, $A_i^\ell \in \mathbb{C}^{n_r \times n_r}$, $B_i^\ell \in \mathbb{C}^{n_r \times 1}$, (A_i^ℓ, B_i^ℓ) be controllable and suppose that $v \neq 0$ and F_i^ℓ satisfy $(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda_i v$, for $i \in \underline{n}$. Then $v \notin \text{Im } B_i^\ell$ for $i \in \underline{n}$.

Proof. Suppose for a contradiction that $v \in \text{Im } B_k^\ell$ for some $k \in \underline{n}$. Then $v = B_k^\ell u$ for some $0 \neq u \in \mathbb{C}$. We have $(A_k^\ell + B_k^\ell F_k^\ell)B_k^\ell u = \lambda_k B_k^\ell u$ and hence B_k^ℓ and $A_k^\ell B_k^\ell$ are linearly dependent, which prevents the pair (A_k^ℓ, B_k^ℓ) from being controllable because $n_r > 1$. □

Lemma 4 establishes that, for a single-input DTSS with controllable subsystems, every vector that can be made a common eigenvector by feedback will not be contained in the image of any input matrix B_i^ℓ . Therefore, in this case such constraint on the optimisation problem is justified.

In the sequel, we will say that a set of matrix pairs is controllable if every matrix pair in the set is controllable. We also require the following definitions.

Definition 1 (CEAS, γ -CEAS). A set of matrix pairs $\mathcal{Z}^\ell = \{(A_i^\ell \in \mathbb{C}^{n_r \times n_r}, B_i^\ell \in \mathbb{C}^{n_r \times 1}) : i \in \underline{n}\}$ is said to be CEAS (Common Eigenvector Assignable with Stability) if there exist $0 \neq v \in \mathbb{C}^{n_r}$, $\lambda_i^\ell \in \mathbb{C}$ with $|\lambda_i^\ell| < 1$, and matrices $F_i^\ell \in \mathbb{C}^{1 \times n_r}$, such that

$$(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda_i^\ell v, \quad \text{for } i \in \underline{n}, \quad (25)$$

If \mathcal{Z}^ℓ is CEAS, we say that $v \in \mathbb{C}^{n_r}$ and $F_i \in \mathbb{C}^{1 \times n_r}$ are compatible with \mathcal{Z}^ℓ whenever (25) holds for $|\lambda_i^\ell| < 1$. If (25) holds with $|\lambda_i| \leq 1 - \gamma$ for some $0 < \gamma \leq 1$, we say that \mathcal{Z}^ℓ is γ -CEAS and refer to the corresponding v and F_i^ℓ as γ -compatible with \mathcal{Z}^ℓ .

Definition 2 (SLASF, γ -SLASF). A set of matrix pairs $\mathcal{Z} = \{(A_i \in \mathbb{C}^{n \times n}, B_i \in \mathbb{C}^{n \times 1}) : i \in \underline{n}\}$ is said to be SLASF (Solvable Lie Algebra with Stability by Feedback) if there exist $K_i \in \mathbb{C}^{1 \times n}$ such that A_i^{cl} as in (4) generate a solvable Lie algebra and satisfy $\rho(A_i^{\text{cl}}) < 1$. If \mathcal{Z} is SLASF, we say that $K_i \in \mathbb{C}^{1 \times n}$ are compatible with \mathcal{Z} if A_i^{cl} as in (4) generate a solvable Lie algebra and satisfy $\rho(A_i^{\text{cl}}) < 1$. If K_i exist so that, in addition, $\rho(A_i^{\text{cl}}) \leq 1 - \gamma$ for some $0 < \gamma \leq 1$, we say that \mathcal{Z} is γ -SLASF and refer to the corresponding K_i as γ -compatible with \mathcal{Z} .

We next state a version of our previous result of [9] for the specific case of single-input systems as follows.

Theorem 1. Let $\mathcal{Z} = \{(A_i \in \mathbb{C}^{n \times n}, B_i \in \mathbb{C}^{n \times 1}) : i \in \underline{n}\}$, consider Algorithm 1 for some suitable choice of ϵ_c^ℓ and ϵ_d^ℓ , and let $\mathcal{Z}^\ell = \{(A_i^\ell, B_i^\ell) : i \in \underline{n}\}$ and $0 < \gamma \leq 1$. Then, \mathcal{Z} is γ -SLASF if and only if \mathcal{Z}^ℓ is γ -CEAS and v_1^ℓ, F_i^ℓ are γ -compatible with \mathcal{Z}^ℓ for $\ell = 1, \dots, n$.

Suppose that $\hat{\mathcal{Z}}^\ell = \{(\hat{A}_i^\ell, \hat{B}_i^\ell) : i \in \underline{n}\}$ is CEAS. Whenever $\hat{F}_i^\ell, \hat{v}_1^\ell$ are compatible with $\hat{\mathcal{Z}}^\ell$ and a unitary matrix having \hat{v}_1^ℓ as its first column is given:

$$[\hat{v}_1^\ell | \hat{v}_2^\ell | \dots | \hat{v}_{n-\ell+1}^\ell], \quad (26)$$

then $\hat{A}_i^{\ell, \text{cl}}, \hat{U}_{\ell+1}, \hat{A}_i^{\ell+1}$ and $\hat{B}_i^{\ell+1}$ will denote the matrices given by (7), (10), (11) and (12), respectively, when the hatted matrices are employed.

The main result of this section, namely Theorem 3, requires the following preliminary theorem.

Theorem 2. Let $\hat{\mathcal{Z}}^\ell = \{(\hat{A}_i^\ell, \hat{B}_i^\ell) : i \in \underline{n}\}$ be γ -CEAS and controllable, with $\hat{A}_i^\ell \in \mathbb{C}^{n_r \times n_r}, \hat{B}_i^\ell \in \mathbb{C}^{n_r \times 1}, n_r > 1, 0 < \gamma \leq 1$. Consider the following sets

$$\hat{\mathcal{T}}_{i,\gamma}^\ell = \{v \in \mathcal{S}_1 : \exists F \in \mathbb{C}^{1 \times n_r}, \lambda \in \mathbb{C} \text{ such that } (\hat{A}_i^\ell + \hat{B}_i^\ell F)v = \lambda v \text{ with } |\lambda| \leq 1 - \gamma\}, \quad (27)$$

$$\hat{\mathcal{T}}_\gamma^\ell = \bigcap_{i=1}^n \hat{\mathcal{T}}_{i,\gamma}^\ell, \quad (28)$$

and the following quantity

$$\epsilon_d^{\ell,*} := \inf_{v \in \hat{\mathcal{T}}_\gamma^\ell} \min_{i \in \underline{n}} d(v, \text{Im } \hat{B}_i^\ell). \quad (29)$$

Then, $\epsilon_d^{\ell,*} > 0$ and each $0 < \epsilon_c^\ell < \gamma$ and $0 < \epsilon_d^\ell < \epsilon_d^{\ell,*}$ ensure that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ so that for each A_i^ℓ, B_i^ℓ satisfying

$$\begin{cases} \|\hat{A}_i^\ell - A_i^\ell\| < \delta, \\ \|\hat{B}_i^\ell - B_i^\ell\| < \delta, \end{cases}$$

there exist $\hat{v}_1^\ell \in \hat{\mathcal{S}}(\epsilon_c^\ell/2, \epsilon_d^\ell/2)$ and \hat{F}_i^ℓ compatible with $\hat{\mathcal{Z}}^\ell$ ($\hat{v}_1^\ell, \hat{F}_i^\ell$ may depend on the specific A_i^ℓ, B_i^ℓ), and a unitary matrix (26) that cause

i)

$$\|\hat{v}_1^\ell - v_1^\ell\| < \epsilon, \quad (30)$$

$$\|\hat{F}_i^\ell - F_i^\ell\| < \epsilon, \quad (31)$$

where v_1^ℓ and F_i^ℓ are the output of Procedure CEA with A_i^ℓ , B_i^ℓ , $\epsilon_c = \epsilon_c^\ell$, and $\epsilon_d = \epsilon_d^\ell$ as inputs.

ii)

$$\begin{aligned} \|\hat{U}_{\ell+1} - U_{\ell+1}\| &< \epsilon, \\ \|\hat{A}_i^{\ell+1} - A_i^{\ell+1}\| &< \epsilon, \end{aligned} \quad (32)$$

$$\|\hat{B}_i^{\ell+1} - B_i^{\ell+1}\| < \epsilon, \quad (33)$$

where $U_{\ell+1}$, $A_i^{\ell+1}$ and $B_i^{\ell+1}$ are the matrices computed at iteration ℓ of Algorithm 1 from A_i^ℓ and B_i^ℓ , with v_1^ℓ and F_i^ℓ as above.

Theorem 2 i) establishes that if the matrices A_i^ℓ and B_i^ℓ given as inputs to Procedure CEA are sufficiently close to some \hat{A}_i^ℓ and \hat{B}_i^ℓ which form a CEAS set $\hat{\mathcal{Z}}^\ell$, then the vector v_1^ℓ and feedback matrices F_i^ℓ computed by such procedure will be as close as desired to some \hat{v}_1^ℓ and \hat{F}_i^ℓ compatible with $\hat{\mathcal{Z}}^\ell$. In general, whether the (given) matrices A_i^ℓ , B_i^ℓ are sufficiently close to some \hat{A}_i^ℓ , \hat{B}_i^ℓ with the required property will not be known. However, the significance of this result lies precisely in the fact that it establishes a type of continuity relation between the result of the procedure and an “exact” result, even if the latter result is not known. In addition, such continuity justifies the numerical implementation of the procedure, since numerical computation will always yield an approximate result.

In broad terms, Theorem 2 ii) shows that if, at step ℓ of Algorithm 1, the matrices A_i^ℓ and B_i^ℓ are sufficiently close to some “exact” ones, then the same will happen at step $\ell + 1$. The proof of Theorem 2 is highly non-trivial and given in Appendix A.1.

We are now ready to state the main result of the paper.

Theorem 3. Let $\hat{\mathcal{Z}} = \{(\hat{A}_i \in \mathbb{C}^{n \times n}, \hat{B}_i \in \mathbb{C}^{n \times 1}) : i \in \underline{n}\}$ be SLASF and controllable. Then, there exist $\epsilon_c^*, \epsilon_d^* > 0$ such that each $0 < \epsilon_c < \epsilon_c^*$ and $0 < \epsilon_d < \epsilon_d^*$ ensure that

i) For every $\epsilon > 0$ there exists a corresponding $\delta > 0$ so that for each A_i, B_i satisfying

$$\|\hat{A}_i - A_i\| < \delta, \quad (34)$$

$$\|\hat{B}_i - B_i\| < \delta, \quad (35)$$

there exist \hat{K}_i compatible with $\hat{\mathcal{Z}}$ (\hat{K}_i may depend on the specific A_i, B_i) that cause $\rho(\hat{A}_i^{\text{cl}}) \leq 1 - \epsilon_c/2$ and

$$\|\hat{A}_i^{\text{cl}} - A_i^{\text{cl}}\| < \epsilon, \quad (36)$$

where $\hat{A}_i^{\text{cl}} = \hat{A}_i + \hat{B}_i \hat{K}_i$, and A_i^{cl} satisfies (4) with K_i obtained as output of Algorithm 1 with A_i, B_i, ϵ_c , and ϵ_d as inputs.

ii) There exists $\epsilon > 0$ for which the closed-loop DTSS (3)–(4) admits a CQLF, provided (34)–(35) are satisfied with δ corresponding to ϵ as in i) above.

Theorem 3 establishes that Algorithm 1 will compute suitable feedback matrices not only in the “exact” case when the given A_i , B_i form a SLASF set, but also when they are close to other (possibly unknown) matrices \hat{A}_i , \hat{B}_i with such property. This justifies the use of Algorithm 1 for control design, since it gives a kind of robustness result for the feedback matrices computed by the algorithm. Theorem 3 also establishes that suitable feedback matrices will be computed for all positive ϵ_c and ϵ_d respectively less than ϵ_c^* and ϵ_d^* . The latter quantities may be not known, since they depend on the possibly unknown \hat{A}_i and \hat{B}_i . Consequently, in theory the parameters ϵ_c and ϵ_d should be selected as small as computationally possible. In practice, however, there is a tradeoff in the selection of ϵ_c since the smaller ϵ_c , the higher the chances of Algorithm 1 yielding unsuitable feedback matrices when more than one common eigenvector could be assigned by feedback with one of them having unstable corresponding eigenvalues and another having stable ones (Procedure CEA could select a local optimiser at the boundary of the constraint set instead of a global optimiser in its interior).

After executing Algorithm 1 to compute feedback matrices K_i , we can check whether the closed-loop DTSS (3)–(4) admits a CQLF by solving the following LMIs

$$P = P^T > 0, \quad P - (A_i^{\text{cl}})^T P A_i^{\text{cl}} > 0, \quad \text{for } i \in \underline{n}. \quad (37)$$

Note that, in this case, LMIs are used only to check whether a CQLF for the closed-loop system exists, and not for feedback design. If these LMIs are feasible, then not only the closed-loop DTSS admits a CQLF but also we have structural information on the DTSS since K_i are such that the A_i^{cl} are suitably close to being simultaneously triangularisable.

4 Examples

We next provide some numerical examples to illustrate the advantages and limitations of the proposed feedback design strategy. For the numerical implementation of Procedure CEA, a feasible vector is first sought using MATLAB[®] OPTIMIZATION TOOLBOX function `fgoalattain`. If such vector is found, it is passed as initial point to the optimisation, implemented via the function `fmincon`.

4.1 Randomly created DTSS

The following subsystems were created randomly but such that $\rho(A_1) < 1$ and $\rho(A_2^{-1}) < 1$.

$$A_1 = \begin{bmatrix} 0.574 & 0.074 & 0.089 \\ 0.074 & 0.572 & -0.091 \\ 0.089 & -0.091 & 0.538 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.038 \\ 0.327 \\ 0.175 \end{bmatrix}, \quad (38)$$

$$A_2 = \begin{bmatrix} -0.737 & 0.386 & -1.680 \\ 1.351 & 0.638 & 0.035 \\ 1.071 & -1.295 & -0.936 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.114 \\ 1.067 \end{bmatrix}. \quad (39)$$

Executing Algorithm 1 choosing $\epsilon_c = \epsilon_d = 10^{-4}$ yields a feasible optimisation at every iteration and returns

$$\begin{aligned} K_1 &= [-3.6480 \quad -7.2304 \quad 8.7751], & U &= \begin{bmatrix} 0.4647 & 0.8287 & -0.3120 \\ -0.7770 & 0.2126 & -0.5925 \\ -0.4246 & 0.5178 & 0.7427 \end{bmatrix}. \\ K_2 &= [-0.3159 \quad 2.0235 \quad 0.2695], \end{aligned} \quad (40)$$

It can be shown that LMIs (37) are feasible and hence the closed-loop DTSS (3)–(4) with K_i as in (40) admits a CQLF and is hence stable under arbitrary switching. In this case, it can be checked that $U^* A_i^{\text{cl}} U$ are not upper triangular but are close in the sense that the entries below the main diagonal are

small. Therefore, the use of feedback matrices (40) designed via Algorithm 1 provides some insight into the structure of the closed-loop DTSS.

On the other hand, solution of the LMIs (5) for both feedback design and CQLF computation, which yields

$$K_1 = [-1.2267 \ -0.7211 \ -1.8731], \quad K_2 = [-0.5140 \ 1.3826 \ 1.1613],$$

is guaranteed to produce a closed-loop DTSS stable under arbitrary switching but provides no structural information.

4.2 DTSS with no CQLF

Consider the systems

$$A_1 = \begin{bmatrix} 0.5 & \alpha \\ 0 & 0.5 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0 \\ \alpha & 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $\alpha = 1.5$, the LMIs (5) are not feasible. Therefore, no CQLF exists for this DTSS. Executing Algorithm 1 with $\epsilon_c = \epsilon_d = 10^{-4}$ yields an infeasible optimisation at the first iteration, $\mathcal{S}(\epsilon_c, \epsilon_d) = \emptyset$.

For $\alpha = 1.4999$, the LMIs (5) are feasible. In this case, Algorithm 1 for the selected ϵ_c, ϵ_d yields an infeasible optimisation. However, reducing ϵ_c to 10^{-5} allows the algorithm again to compute suitable feedback matrices.

4.3 CQLF exists but Algorithm 1 fails

Consider again the DTSS (38)–(39), with the addition of the subsystem

$$A_3 = \begin{bmatrix} 0.352 & 0.159 & -1.129 \\ 0.159 & 0 & 0.262 \\ -1.129 & 0.262 & -0.705 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0.433 \\ 0 \\ 0 \end{bmatrix}.$$

Algorithm 1 for $\epsilon_c = \epsilon_d = 10^{-4}$ yields

$$\begin{aligned} K_1 &= [-15.3542 \ 3.8969 \ -11.3814], \\ K_2 &= [0.0734 \ 0.9747 \ 2.7288], \\ K_3 &= [-1.3542 \ 0.8334 \ -4.5001], \\ U &= \begin{bmatrix} 0.1662+0.0234j & -0.0918-0.6508j & -0.7347 \\ 0.9744+0.1374j & 0.0093+0.0662j & 0.1650 \\ 0.0587+0.0083j & 0.1049+0.7433j & -0.6580 \end{bmatrix}. \end{aligned} \tag{41}$$

However, the optimisation informs that there is an active inequality, corresponding to the stability constraint (18). In this case, the LMIs (37) are not feasible and hence no CQLF exists when the feedback matrices (41) are employed.

On the other hand, the LMIs (5) are feasible and hence other feedback matrices may indeed produce a closed-loop DTSS with a CQLF.

5 Conclusions

This paper complements the theoretical results in [9], and contributes to furthering the understanding of fundamental system structure in feedback stabilisation of DTSS. We have presented a numerical

implementation of a feedback design strategy for DTSS based on Lie-algebraic solvability. The proposed strategy seeks feedback matrices to achieve a closed-loop system structure that *approximates* that required to satisfy such Lie-algebraic stability criteria.

The main theoretical contribution of the paper establishes that if a system for which the Lie-algebraic conditions considered exists in a suitably small neighbourhood of the given system data, then our implementation will find feedback matrices so that the corresponding closed-loop DTSS admits a CQLF *even if the considered Lie-algebraic conditions are not met by the given system*. However, since the existence of such feasible “exact” system is in general unknown, the resulting closed-loop system is not guaranteed to admit a CQLF but the latter may be checked with a set of *informed* LMIs built with the computed closed-loop matrices. Whether such “exact” system exists suitably close to a given system has not been discussed, and remains a topic for further research. Future work will also consider extensions to multiple input systems, which are nontrivial and will possibly require the consideration of controllability indices in the algorithm.

A Appendix

A.1 Proof of Theorem 2

Throughout this proof, a hatted expression denotes the expression given by the corresponding equations when matrices $\hat{A}_i^\ell, \hat{B}_i^\ell$ are substituted for A_i^ℓ, B_i^ℓ . For example, $\hat{H}_i(v) = (v v^* - I) \hat{B}_i^\ell$.

We begin by establishing that $\epsilon_d^{\ell,*} > 0$. The set $\hat{\mathcal{T}}_\gamma^\ell$ is the set of all unit vectors that are γ -compatible with $\hat{\mathcal{Z}}^\ell$, and since $\hat{\mathcal{Z}}^\ell$ is γ -CEAS, then $\hat{\mathcal{T}}_\gamma^\ell \neq \emptyset$. Since $\hat{\mathcal{Z}}^\ell$ is controllable, then $(\hat{A}_i^\ell, \hat{B}_i^\ell)$ is controllable and hence $\hat{B}_i^\ell \neq 0$.

Claim 1. *The set $\hat{\mathcal{T}}_\gamma^\ell$ is compact.*

Proof. The set $\hat{\mathcal{T}}_{i,\gamma}^\ell$ is bounded since $\hat{\mathcal{T}}_{i,\gamma}^\ell \subset \mathcal{S}_1$ by definition. The set $\hat{\mathcal{T}}_{i,\gamma}^\ell$ can be equivalently defined as

$$\hat{\mathcal{T}}_{i,\gamma}^\ell = \{v \in \mathcal{S}_1 : \exists \lambda \in \mathbb{C} \text{ such that } \hat{P}_i^\ell(\lambda I - \hat{A}_i^\ell)v = 0, \text{ with } |\lambda| \leq 1 - \gamma\}, \quad (42)$$

where we have defined

$$\hat{P}_i^\ell = \left[I - \hat{B}_i^\ell \left((\hat{B}_i^\ell)^* \hat{B}_i^\ell \right)^{-1} (\hat{B}_i^\ell)^* \right]. \quad (43)$$

Note that since $0 \neq \hat{B}_i^\ell \in \mathbb{C}^{n_r \times 1}$, then \hat{P}_i^ℓ is well-defined. Consider a sequence $\{v_k\}_{k=0}^\infty$ such that $v_k \in \hat{\mathcal{T}}_{i,\gamma}^\ell$ for all $k \geq 0$ and $\lim_{k \rightarrow \infty} v_k = v$. Since $\|v_k\| = 1$ for all $k \geq 0$ and by the continuity of norms, then $\|v\| = 1$ necessarily. For every $k \geq 0$, we have

$$\hat{P}_i^\ell(\lambda_k I - \hat{A}_i^\ell)v_k = 0, \quad (44)$$

for some $\lambda_k \in \mathbb{C}$ with $|\lambda_k| \leq 1 - \gamma$. From (44),

$$\lim_{k \rightarrow \infty} \hat{P}_i^\ell v_k \lambda_k = \lim_{k \rightarrow \infty} \left[\hat{P}_i^\ell(v_k - v)\lambda_k + \hat{P}_i^\ell v \lambda_k \right] = \hat{P}_i^\ell \hat{A}_i^\ell v. \quad (45)$$

Since λ_k is bounded and $\lim_{k \rightarrow \infty} v_k = v$, it follows that

$$\lim_{k \rightarrow \infty} \hat{P}_i^\ell v \lambda_k = \hat{P}_i^\ell \hat{A}_i^\ell v. \quad (46)$$

If $\hat{P}_i^\ell v \neq 0$, then $\lim_{k \rightarrow \infty} \lambda_k = \lambda$ with $|\lambda| \leq 1 - \gamma$. If $\hat{P}_i^\ell v = 0$, then $\hat{P}_i^\ell \hat{A}_i^\ell v = 0$. In either case, $v \in \hat{\mathcal{T}}_{i,\gamma}^\ell$ and hence $\hat{\mathcal{T}}_{i,\gamma}^\ell$ is closed. Therefore, $\hat{\mathcal{T}}_\gamma^\ell$ is closed since it is the intersection of a finite number of closed sets. \square

The quantity $\epsilon_d^{\ell,*}$ as defined in (29) is the infimum, over all vectors γ -compatible with $\hat{\mathcal{Z}}^\ell$, of the minimum of the distance between such vectors and $\text{Im } \hat{B}_i^\ell$. Since $(\hat{A}_i^\ell, \hat{B}_i^\ell)$ is controllable and $n_r > 1$, then Lemma 4 implies that $v \notin \text{Im } \hat{B}_i^\ell$ for $i \in \underline{n}$ and every $v \in \hat{\mathcal{T}}_\gamma^\ell$. Therefore, for every $v \in \hat{\mathcal{T}}_\gamma^\ell$, we have $\min_{i \in \underline{n}} d(v, \text{Im } \hat{B}_i^\ell) > 0$ because $\text{Im } \hat{B}_i^\ell$ is closed for $i \in \underline{n}$. Since $\hat{\mathcal{T}}_\gamma^\ell$ is compact and $\min_{i \in \underline{n}} d(v, \text{Im } \hat{B}_i^\ell)$ is continuous and positive at every $v \in \hat{\mathcal{T}}_\gamma^\ell$, it follows that $\min_{i \in \underline{n}} d(v, \text{Im } \hat{B}_i^\ell)$ achieves a minimum on $\hat{\mathcal{T}}_\gamma^\ell$ and hence $\epsilon_d^{\ell,*} > 0$.

A.1.1 Proof of Theorem 2 i)

Claim 2. For every $0 < \epsilon_c^\ell < \gamma$ and $0 < \epsilon_d^\ell < \epsilon_d^{\ell,*}$, there exists $\delta_2 > 0$ such that $\hat{\mathcal{T}}_\gamma^\ell \cap \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell) \neq \emptyset$ for all A_i^ℓ, B_i^ℓ satisfying

$$\|\hat{A}_i^\ell - A_i^\ell\| < \delta_2, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_2. \quad (47)$$

Proof. Let \hat{v}^\natural and \hat{G}_i be γ -compatible with $\hat{\mathcal{Z}}^\ell$ and $\|\hat{v}^\natural\| = 1$. Note that $\hat{v}^\natural \in \hat{\mathcal{T}}_\gamma^\ell$. By (29), \hat{G}_i and \hat{v}^\natural satisfy, for $i \in \underline{n}$,

$$(\hat{A}_i^\ell + \hat{B}_i^\ell \hat{G}_i) \hat{v}^\natural = \lambda_i \hat{v}^\natural, \quad (48)$$

$$|\lambda_i| \leq 1 - \gamma < 1, \quad (49)$$

$$d(\hat{v}^\natural, \text{Im } \hat{B}_i^\ell) \geq \epsilon_d^{\ell,*} > 0. \quad (50)$$

Consider $\hat{H}_i(\hat{v}^\natural)$ and $H_i(\hat{v}^\natural)$ from (14). By (50) and since \hat{B}_i^ℓ have full rank, then $\hat{H}_i(\hat{v}^\natural)$ has full rank. Then, for all $\delta_0 > 0$ sufficiently small, $H_i(\hat{v}^\natural)$ has full rank whenever $\|\hat{B}_i^\ell - B_i^\ell\| < \delta_0$. Whenever the latter holds, the expression (16) is continuous on the entries of A_i^ℓ and B_i^ℓ . Then, given $\epsilon_0 > 0$, we can find $0 < \delta_1 \leq \delta_0$ such that

$$\begin{aligned} & \|\hat{A}_i^{\ell,\text{cl}}(\hat{v}^\natural) \hat{v}^\natural - A_i^{\ell,\text{cl}}(\hat{v}^\natural) \hat{v}^\natural\| < \epsilon_0 \\ \text{whenever } & \begin{cases} \|\hat{A}_i^\ell - A_i^\ell\| < \delta_1, \\ \|\hat{B}_i^\ell - B_i^\ell\| < \delta_1. \end{cases} \end{aligned} \quad (51)$$

By (48) and Lemma 3-iii), we have $\hat{A}_i^{\ell,\text{cl}}(\hat{v}^\natural) \hat{v}^\natural = \lambda_i \hat{v}^\natural$. Since $\|\hat{v}^\natural\| = 1$, then

$$\|\hat{A}_i^{\ell,\text{cl}}(\hat{v}^\natural) \hat{v}^\natural\| = |\lambda_i| \leq 1 - \gamma < 1$$

by (49), and we can select $\epsilon_0 > 0$ small enough so that $\|\hat{A}_i^{\ell,\text{cl}}(\hat{v}^\natural) \hat{v}^\natural\| \leq 1 - \epsilon_c^\ell$ and hence $\hat{v}^\natural \in \mathcal{S}_2(\epsilon_c^\ell)$ whenever (51) holds. For $0 \leq a < \delta_1$, define

$$\underline{d}(a) := \min_{i \in \underline{n}} \inf_{B_i^\ell: \|\hat{B}_i^\ell - B_i^\ell\| \leq a} d(\hat{v}^\natural, \text{Im } B_i^\ell)$$

and note that, by the continuity of $d(\hat{v}^\natural, \text{Im } B_i^\ell)$ on the entries of B_i^ℓ whenever B_i^ℓ has full rank, and since by (50) $\underline{d}(0) \geq \epsilon_d^{\ell,*} > 0$, there exists $\delta_2 > 0$ sufficiently small for which $\underline{d}(\delta_2) > \epsilon_d^\ell > 0$. Therefore, for such δ_2 we have $\hat{v}^\natural \in \mathcal{S}_3(\epsilon_d^\ell)$ and hence $\hat{v}^\natural \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ whenever (47) holds. \square

Claim 3. Consider $0 < \epsilon_c^\ell < \gamma$ and $0 < \epsilon_d^\ell < \epsilon_d^{\ell,*}$. Then, there exists $\delta_3 > 0$ so that

i) (52)–(53) hold for every A_i^ℓ, B_i^ℓ satisfying (54).

$$\sup_{v \in S(\epsilon_c^\ell, \epsilon_d^\ell)} \max_{i \in \underline{N}} \|\hat{A}_i^{\ell, \text{cl}}(v)v\| \leq 1 - \epsilon_c^\ell/2, \quad (52)$$

$$\inf_{v \in S(\epsilon_c^\ell, \epsilon_d^\ell)} \min_{i \in \underline{N}} d(v, \text{Im } \hat{B}_i^\ell) \geq \epsilon_d^\ell/2, \quad (53)$$

$$\|\hat{A}_i^\ell - A_i^\ell\| < \delta_3, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_3. \quad (54)$$

ii) For every $\epsilon_2 > 0$, there exists $0 < \delta_4 < \delta_3$ such that (55) holds for every A_i^ℓ, B_i^ℓ satisfying (56).

$$|\hat{J}(v) - J(v)| < \epsilon_2, \quad \text{for all } v \in S(\epsilon_c^\ell, \epsilon_d^\ell) \quad (55)$$

$$\|\hat{A}_i^\ell - A_i^\ell\| < \delta_4, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_4. \quad (56)$$

Proof. i) Consider the functions

$$f(\{A_i^\ell, B_i^\ell : i \in \underline{N}\}) := \sup_{v \in S(\epsilon_c^\ell, \epsilon_d^\ell)} \max_{i \in \underline{N}} \|\hat{A}_i^{\ell, \text{cl}}(v)v - A_i^{\ell, \text{cl}}(v)v\|,$$

$$g(\{A_i^\ell, B_i^\ell : i \in \underline{N}\}) := \sup_{v \in S(\epsilon_c^\ell, \epsilon_d^\ell)} \max_{i \in \underline{N}} |d(v, \text{Im } \hat{B}_i^\ell) - d(v, \text{Im } B_i^\ell)|,$$

which are non-negative whenever well-defined. Note that

$$f(\{\hat{A}_i^\ell, \hat{B}_i^\ell : i \in \underline{N}\}) = 0 = g(\{\hat{A}_i^\ell, \hat{B}_i^\ell : i \in \underline{N}\}),$$

and that, since \hat{B}_i^ℓ full column rank, f, g are continuous on the entries of A_i^ℓ, B_i^ℓ and well-defined whenever (54) holds for some δ_3 sufficiently small. Therefore, for every $\epsilon_1 > 0$, we can find a corresponding $\delta_3 > 0$ so that

$$f(\{A_i^\ell, B_i^\ell : i \in \underline{N}\}) \leq \epsilon_1, \quad g(\{A_i^\ell, B_i^\ell : i \in \underline{N}\}) \leq \epsilon_1$$

whenever (54) holds. Select $\epsilon_1 \leq \min\{\epsilon_c^\ell/2, \epsilon_d^\ell/2\}$, and take the corresponding δ_3 . Thus, (52)–(53) hold whenever (54) is true.

ii) Consider the function

$$h(\{A_i^\ell, B_i^\ell : i \in \underline{N}\}) := \sup_{v \in S(\epsilon_c^\ell, \epsilon_d^\ell)} |\hat{J}(v) - J(v)|.$$

Note that, because of (53), h is well-defined, non-negative and continuous whenever (54) holds and that

$$h(\{\hat{A}_i^\ell, \hat{B}_i^\ell : i \in \underline{N}\}) = 0.$$

Therefore, given $\epsilon_2 > 0$ we can find $0 < \delta_4 < \delta_3$, such that (55) holds whenever (56) is true. \square

Fix ϵ_c^ℓ and ϵ_d^ℓ so that $0 < \epsilon_c^\ell < \gamma$ and $0 < \epsilon_d^\ell < \epsilon_d^{\ell,*}$. Consider A_i^ℓ and B_i^ℓ satisfying both (47) with δ_2 as given by Claim 2 and (54) with δ_3 from Claim 3-i). Let $v_1^\ell \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ be a (global) minimiser of $J(v)$ subject to $v \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ [such minimiser exists because J is continuous on $\mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ and $\mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ is compact and nonempty]. Define $\hat{\epsilon}_c := \epsilon_c^\ell/2$ and $\hat{\epsilon}_d := \epsilon_d^\ell/2$, pick $\hat{v}^\natural \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell) \cap \hat{\mathcal{T}}_\gamma^\ell$, and note that

$$0 < \hat{\epsilon}_c \leq 1 - \max_{i \in \mathbf{N}} \max \left\{ \|\hat{A}_i^{\ell, \text{cl}}(v_1^\ell) v_1^\ell\|, \|\hat{A}_i^{\ell, \text{cl}}(\hat{v}^\natural) \hat{v}^\natural\| \right\}, \text{ and}$$

$$0 < \hat{\epsilon}_d \leq \min_{i \in \mathbf{N}} \min \left\{ d(v_1^\ell, \text{Im } \hat{B}_i^\ell), d(\hat{v}^\natural, \text{Im } \hat{B}_i^\ell) \right\},$$

by (52)–(53) from Claim 3-i), and since $v_1^\ell \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ and $\hat{v}^\natural \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$. Thus, we have $v_1^\ell \in \hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d)$, $\hat{v}^\natural \in \hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d)$ and by Lemma 3-ii), we know $\hat{J}(\hat{v}^\natural) = 0$. Since v_1^ℓ is a minimiser within $\mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$, and $\hat{v}^\natural \in \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$, then $J(v_1^\ell) \leq J(\hat{v}^\natural)$. Thus, using Claim 3-ii), we have

$$J(v_1^\ell) \leq J(\hat{v}^\natural) = |\hat{J}(\hat{v}^\natural) - J(\hat{v}^\natural)| < \epsilon_2, \quad (57)$$

$$\text{whenever } \|\hat{A}_i^\ell - A_i^\ell\| < \delta_4, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_4. \quad (58)$$

Also, using (57) and Claim 3-ii), under (58) we will have

$$\hat{J}(v_1^\ell) \leq |\hat{J}(v_1^\ell) - J(v_1^\ell)| + J(v_1^\ell) < 2\epsilon_2.$$

Define $\hat{\mathcal{V}}_0 := \{v \in \hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d) \cap \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell) : \hat{J}(v) = 0\}$ and note that $\hat{v}^\natural \in \hat{\mathcal{V}}_0$. For each $\epsilon_3 > 0$, consider

$$\mathcal{B}(\epsilon_3) := \{v \in \hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d) \cap \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell) : d(v, \hat{\mathcal{V}}_0) < \epsilon_3\}.$$

By continuity of J and since both $\hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d)$ and $\mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ are compact, for every $\epsilon_3 > 0$ we can find $\epsilon_2 > 0$ such that

$$\hat{J}(v_1^\ell) < 2\epsilon_2 \Rightarrow v_1^\ell \in \mathcal{B}(\epsilon_3) \quad (59)$$

Note that the closedness of $\hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d)$ and $\mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$, and hence of $\hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d) \cap \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$, is key in allowing the implication (59). $v_1^\ell \in \mathcal{B}(\epsilon_3)$ implies the existence of $\hat{v}_1^\ell \in \hat{\mathcal{V}}_0$ such that $\|\hat{v}_1^\ell - v_1^\ell\| < \epsilon_3$, and $\hat{v}_1^\ell \in \hat{\mathcal{V}}_0$ implies that $\hat{J}(\hat{v}_1^\ell) = 0$. By Lemma 3 and since $\hat{v}_1^\ell \in \hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d)$, we have $\hat{A}_i^{\ell, \text{cl}}(\hat{v}_1^\ell) \hat{v}_1^\ell = \alpha_i \hat{v}_1^\ell$ with $|\alpha_i| \leq 1 - \hat{\epsilon}_c < 1$, which establishes that \hat{v}_1^ℓ and $\hat{F}_i^\ell := \hat{M}_i(\hat{v}_1^\ell)$ are compatible with $\hat{\mathcal{Z}}^\ell$.

Next, since $H_i(v)$ has full rank whenever $v \in \hat{\mathcal{S}}(\hat{\epsilon}_c, \hat{\epsilon}_d) \cap \mathcal{S}(\epsilon_c^\ell, \epsilon_d^\ell)$ and (58), then $M_i(v)$ is continuous on v and on the entries of A_i^ℓ, B_i^ℓ . Therefore, given $\epsilon > 0$ we can find $\delta_5 > 0$ such that $\|\hat{M}_i(\hat{v}_1^\ell) - M_i(v_1^\ell)\| < \epsilon$ whenever

$$\|\hat{A}_i^\ell - A_i^\ell\| < \delta_5, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_5, \quad \|\hat{v}_1^\ell - v_1^\ell\| < \delta_5.$$

Take $\epsilon_3 = \min\{\epsilon, \delta_5\}$ to select δ_4 as above. Finally, taking $\delta = \min\{\delta_2, \delta_4, \delta_5\}$ concludes the proof of Theorem 2 i).

A.1.2 Proof of Theorem 2 ii)

Since (11) is continuous on the entries of $U_{\ell+1}$ and $A_i^{\ell, \text{cl}}$, given $\epsilon > 0$ we can find $0 < \delta_1 < \epsilon$ such that (32) holds whenever

$$\begin{aligned} \|\hat{A}_i^{\ell, \text{cl}} - A_i^{\ell, \text{cl}}\| &< \delta_1, \\ \|\hat{U}_{\ell+1} - U_{\ell+1}\| &< \delta_1. \end{aligned} \quad (60)$$

Since (12) is continuous on the entries of B_i^ℓ and $U_{\ell+1}$, then given $\epsilon > 0$ we can find $0 < \delta_2 < \delta_1$ so that (33) holds whenever

$$\begin{aligned} \|\hat{B}_i^\ell - B_i^\ell\| &< \delta_2, \\ \|\hat{U}_{\ell+1} - U_{\ell+1}\| &< \delta_2. \end{aligned} \quad (61)$$

Consider square unitary matrices $\hat{W} = [\hat{v}_1^\ell | \hat{U}_{\ell+1}]$ and $W = [v_1^\ell | U_{\ell+1}]$. Note that, given \hat{v}_1^ℓ , v_1^ℓ and $U_{\ell+1}$ so that W is square and unitary, and $\|\hat{v}_1^\ell\| = 1$, for every $\delta_2 > 0$ we can find $\hat{U}_{\ell+1}$ and $0 < \delta_3 < \delta_2$ so that

$$\|\hat{W} - W\| < \delta_2 \quad (62)$$

whenever $\|\hat{v}_1^\ell - v_1^\ell\| < \delta_3$. Note that (62) implies (61) since $\hat{U}_{\ell+1}$ and $U_{\ell+1}$ are the last $n - \ell$ columns of \hat{W} and W , respectively. Since (7) is continuous, given $\delta_1 > 0$ we can find $0 < \delta_4 < \delta_3$ so that (60) holds whenever

$$\begin{aligned} \|\hat{A}_i^\ell - A_i^\ell\| &< \delta_4, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_4, \quad \text{and} \\ \|\hat{F}_i^\ell - F_i^\ell\| &< \delta_4. \end{aligned} \quad (63)$$

Applying Theorem 2 i) yields that each $0 < \epsilon_c^\ell < \gamma$, $0 < \epsilon_d^\ell < \epsilon_d^{\ell,*}$ ensure that for the given $\delta_4 > 0$, we can find $0 < \delta < \delta_4$ so that for each A_i^ℓ, B_i^ℓ satisfying $\|\hat{A}_i^\ell - A_i^\ell\| < \delta$ and $\|\hat{B}_i^\ell - B_i^\ell\| < \delta$, there exist $\hat{v}_1^\ell \in \hat{\mathcal{S}}(\epsilon_c^\ell/2, \epsilon_d^\ell/2)$ and \hat{F}_i^ℓ compatible with $\hat{\mathcal{Z}}^\ell$ so that (63) holds and $\|\hat{v}_1^\ell - v_1^\ell\| < \delta_4 < \epsilon$.

A.2 Proof of Theorem 3

i) Since $\hat{\mathcal{Z}}$ is SLASF, then it is γ -SLASF for some $0 < \gamma \leq 1$. Then, Theorem 1 shows that $\hat{\mathcal{Z}}^\ell$ is γ -CEAS for $\ell = 1, \dots, n$, irrespective of which \hat{v}_1^ℓ and \hat{F}_i^ℓ γ -compatible with $\hat{\mathcal{Z}}^\ell$ are taken at each iteration ℓ . Since $\hat{\mathcal{Z}}$ is controllable, then $(\hat{A}_i^1 = \hat{A}_i, \hat{B}_i^1 = \hat{B}_i)$ is controllable and $(\hat{A}_i^\ell, \hat{B}_i^\ell)$ is controllable for $\ell = 1, \dots, n$ (see, for example, Proposition 1.2 of [14]).

Since (\hat{A}_i, \hat{B}_i) is controllable and $\hat{B}_i \in \mathbb{C}^{n \times 1}$, then $\hat{B}_i \neq 0$ and \hat{B}_i has full column rank. By the continuity of (4), given $\epsilon > 0$ we can find $\epsilon_1 > 0$ such that (36) holds whenever

$$\|\hat{A}_i - A_i\| < \epsilon_1, \quad \|\hat{B}_i - B_i\| < \epsilon_1, \quad (64)$$

$$\|\hat{K}_i - K_i\| < \epsilon_1. \quad (65)$$

By (8), given $\epsilon_1 > 0$ we can find $0 < \epsilon_2 < \epsilon_1$ such that (65) holds whenever

$$\|\hat{F}_i^\ell - F_i^\ell\| < \epsilon_2, \quad \|\hat{U}_\ell - U_\ell\| < \epsilon_2, \quad \text{for } \ell = 1, \dots, n. \quad (66)$$

For $\ell = n$, \hat{A}_i^n and \hat{B}_i^n are scalars and $\hat{B}_i^n \neq 0$ because $(\hat{A}_i^n, \hat{B}_i^n)$ is controllable. Consider the following condition

$$\|\hat{A}_i^n - A_i^n\| < \epsilon_3, \quad \|\hat{B}_i^n - B_i^n\| < \epsilon_3. \quad (67)$$

Note that if we select $0 < \epsilon_3 < \epsilon_2$ small enough then (67) will imply that $B_i^n \neq 0$. Taking such small ϵ_3 , for each set $\{(A_i^n, B_i^n) : i \in \underline{n}\}$ satisfying (67) Procedure CEA in Algorithm 1 does not employ ϵ_d^n , and returns F_i^n such that $|A_i^{n,cl}| \leq 1 - \epsilon_c^n$, which is possible for every $0 < \epsilon_c^n \leq 1$. Note that if we choose

$\epsilon_3 > 0$ small enough, F_i^n will, in addition, satisfy $|\hat{A}_i^n + \hat{B}_i^n F_i^n| \leq 1 - \epsilon_c^n/2$. Therefore, we may take $\hat{F}_i^n = F_i^n$.

If $n = 1$, i) is established taking $\epsilon_c^* = 1$, arbitrary $\epsilon_d^* > 0$, and $\delta = \epsilon_3$, since in this case $K_i = F_i^1$ and by the above argument we may take $\hat{K}_i = K_i$.

We proceed for $n > 1$. We next establish the existence of ϵ_c^* and ϵ_d^* . For $\ell = 1, \dots, n-1$, i.e. for $n_r > 1$, consider the sets $\hat{\mathcal{T}}_\gamma^\ell$ as defined in (27). Note that the set $\hat{\mathcal{T}}_\gamma^\ell$ depends on the matrices \hat{A}_i^ℓ and \hat{B}_i^ℓ , for $i \in \underline{n}$. Since the latter matrices depend on the specific vectors and matrices computed at previous iterations of Algorithm 1, then $\hat{\mathcal{T}}_\gamma^\ell$ also depends on such quantities. For $\ell = 1, \dots, n-1$ and $k = 1, \dots, \ell$, consider the expression

$$\eta_{\gamma}^{k,\ell} := \inf_{\hat{v}_1^k \in \hat{\mathcal{T}}_\gamma^k} \inf_{\hat{v}_1^{k+1} \in \hat{\mathcal{T}}_\gamma^{k+1}} \cdots \left[\inf_{\hat{v}_1^\ell \in \hat{\mathcal{T}}_\gamma^\ell} \min_{i \in \underline{n}} d(\hat{v}_1^\ell, \text{Im } \hat{B}_i^\ell) \right]. \quad (68)$$

Note that the expression between square brackets in (68) coincides with $\epsilon_d^{\ell,*}$ as given in (29). The expressions (68) can be interpreted as the infimum of the minimum distance between all the possible γ -compatible vectors that could be obtained at iteration ℓ and all the possible $\text{Im } \hat{B}_i^\ell$ that can be obtained at iteration ℓ , having the data corresponding to iteration $k \leq \ell$ and over all possible outcomes at iterations $k, k+1, \dots, \ell$. We make the following claim, to be proved later.

Claim 4. For $\ell = 1, \dots, n-1$ and $k = 1, \dots, \ell$, expression (68)

a) may depend on \hat{A}_i^k, \hat{B}_i^k and γ but does not depend on \hat{F}_i^r or \hat{U}_{r+1} for $k \leq r \leq \ell-1$.

b) is positive.

By Claim 4, knowing $\hat{A}_i^1 = \hat{A}_i, \hat{B}_i^1 = \hat{B}_i$ and γ , we can define positive

$$\epsilon_c^* := \gamma, \quad (69)$$

$$\epsilon_d^* := \min_{\ell=1, \dots, n-1} \eta_{\gamma}^{1,\ell}. \quad (70)$$

Note that $\epsilon_d^* \leq \epsilon_d^{\ell,*}$ for $\ell = 1, \dots, n-1$, with $\epsilon_d^{\ell,*}$ as in (29).

Applying Theorem 2 repeatedly ($n-1$ times), it follows that each $0 < \epsilon_c^\ell < \epsilon_c^*$ and $0 < \epsilon_d^\ell < \epsilon_d^*$ ensure that given $\delta_n := \epsilon_3 > 0$, we can find a corresponding $0 < \delta_\ell < \epsilon_3$, for $\ell = n-1, \dots, 1$ so that for each $\hat{A}_i^\ell, \hat{B}_i^\ell$ satisfying

$$\|\hat{A}_i^\ell - A_i^\ell\| < \delta_\ell, \quad \|\hat{B}_i^\ell - B_i^\ell\| < \delta_\ell, \quad (71)$$

there exist¹ $\hat{v}_1^\ell \in \hat{\mathcal{S}}^\ell(\epsilon_c^\ell/2, \epsilon_d^\ell/2)$ and \hat{F}_i^ℓ compatible with $\hat{\mathcal{Z}}^\ell$, and a unitary matrix (26) that cause, for $\ell = 1, \dots, n-1$,

$$\|\hat{F}_i^\ell - F_i^\ell\| < \delta_{\ell+1}, \quad \|\hat{v}_1^\ell - v_1^\ell\| < \delta_{\ell+1}, \quad (72)$$

$$\|\hat{U}_{\ell+1} - U_{\ell+1}\| < \delta_{\ell+1}, \quad \|\hat{A}_i^{\ell+1} - A_i^{\ell+1}\| < \delta_{\ell+1}, \quad \|\hat{B}_i^{\ell+1} - B_i^{\ell+1}\| < \delta_{\ell+1}. \quad (73)$$

By definition, $\hat{U}_1 = U_1 = I$. The latter fact jointly with (72)–(73) establish (66). Take $\delta = \delta_1$. Since $\hat{v}_1^\ell \in \hat{\mathcal{S}}^\ell(\epsilon_c^\ell/2, \epsilon_d^\ell/2)$ and $\hat{v}_1^\ell, \hat{F}_i^\ell$ are compatible with $\hat{\mathcal{Z}}^\ell$ then Lemma 3 shows that $(\hat{A}_i^\ell + \hat{B}_i^\ell \hat{F}_i^\ell) \hat{v}_1^\ell = \lambda_i^\ell \hat{v}_1^\ell$

¹We add a superscript ℓ to the set \mathcal{S} to denote that this set is different at each iteration of Algorithm 1.

with $|\lambda_i^\ell| \leq 1 - \epsilon_c^\ell/2$, for $\ell = 1, \dots, n-1$. Also, λ_i^ℓ is an eigenvalue of \hat{A}_i^{cl} (recall Remark 1), and so is $\hat{A}_i^\ell + \hat{B}_i^\ell \hat{F}_i^\ell$, which satisfies $|\hat{A}_i^\ell + \hat{B}_i^\ell \hat{F}_i^\ell| \leq 1 - \epsilon_c^\ell/2$. Therefore, $\rho(\hat{A}_i^{\text{cl}}) \leq 1 - \epsilon_c/2$ and the proof of part i) is concluded if Claim 4 is true.

To establish Claim 4, consider the sets $\hat{\mathcal{T}}_\gamma^\ell$ defined in (28), for $\ell = 1, \dots, n-1$. Note that $\hat{\mathcal{T}}_\gamma^\ell$ depends on $\hat{A}_i^\ell, \hat{B}_i^\ell$ which, by (7) and (9)–(12), may depend on $\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}, \hat{F}_i^{\ell-1}$ and \hat{U}_ℓ . We next establish that $\hat{\mathcal{T}}_\gamma^\ell$ does not depend on $\hat{F}_i^{\ell-1}$. From (7), (11) and (12), we have $\hat{A}_i^\ell = \hat{U}_\ell^* \hat{A}_i^{\ell-1} \hat{U}_\ell + \hat{B}_i^\ell \hat{F}_i^{\ell-1} \hat{U}_\ell$. Consider \hat{P}_i^ℓ as in (43), and note that $\hat{P}_i^\ell \hat{B}_i^\ell = 0$. Thus, it follows that $\hat{P}_i^\ell (\lambda I - \hat{A}_i^\ell) = \hat{P}_i^\ell (\lambda I - \hat{U}_\ell^* \hat{A}_i^{\ell-1} \hat{U}_\ell)$, which does not depend on $\hat{F}_i^{\ell-1}$. From the equivalent definition of $\hat{\mathcal{T}}_\gamma^\ell$ in (42)–(43), it follows that $\hat{\mathcal{T}}_\gamma^\ell$ does not depend on $\hat{F}_i^{\ell-1}$. Following similar considerations, we can show that $\hat{\mathcal{T}}_\gamma^\ell$ does not depend on \hat{F}_i^k for $k = 1, \dots, \ell-1$.

Next, consider $\eta_\gamma^{\ell,\ell}$ from (68), rewritten to explicitly show the dependence on matrices:

$$\eta_\gamma^{\ell,\ell} = \inf_{\hat{v}^\ell \in \hat{\mathcal{T}}_\gamma^\ell(\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}, \hat{U}_\ell)} \min_{i \in \mathbb{N}} d(\hat{v}^\ell, \text{Im } \hat{U}_\ell^* \hat{B}_i^{\ell-1}), \quad (74)$$

We next show that for fixed $\hat{A}_i^{\ell-1}$ and $\hat{B}_i^{\ell-1}$, (74) depends only on the orthogonal complement of \hat{U}_ℓ and not on \hat{U}_ℓ itself. Consider fixed $\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}, \hat{U}_\ell$, and let $\hat{v}_1^{\ell-1}$ be such that $[\hat{v}_1^{\ell-1} | \hat{U}_\ell]$ is unitary. Let \tilde{U}_ℓ also make $[\hat{v}_1^{\ell-1} | \tilde{U}_\ell]$ unitary. Write $\hat{B}_i^{\ell-1}$ as

$$\hat{B}_i^{\ell-1} = \hat{v}_1^{\ell-1} \alpha + \hat{U}_\ell \hat{\beta}_i = \hat{v}_1^{\ell-1} \alpha + \tilde{U}_\ell \tilde{\beta}_i \quad (75)$$

Using (75) we may write

$$d(\hat{v}^\ell, \text{Im } \hat{U}_\ell^* \hat{B}_i^{\ell-1}) = d(\hat{v}^\ell, \text{Im } \hat{\beta}_i) = d(\hat{v}^\ell, \text{Im } \hat{U}_\ell^* \tilde{U}_\ell \tilde{\beta}_i) \quad (76)$$

Put $\hat{v}^\ell = \hat{U}_\ell^* \tilde{U}_\ell \tilde{v}^\ell$ and, since $\hat{U}_\ell^* \tilde{U}_\ell$ is unitary, we have

$$d(\hat{v}^\ell, \text{Im } \hat{\beta}_i) = d(\tilde{v}^\ell, \text{Im } \tilde{\beta}_i), \quad \text{and} \quad \min_{i \in \mathbb{N}} d(\hat{v}^\ell, \text{Im } \hat{\beta}_i) = \min_{i \in \mathbb{N}} d(\tilde{v}^\ell, \text{Im } \tilde{\beta}_i). \quad (77)$$

Let $\tilde{\mathcal{T}}_\gamma^\ell = \hat{\mathcal{T}}_\gamma^\ell(\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}, \tilde{U}_\ell)$ and note that $\hat{v}^\ell \in \hat{\mathcal{T}}_\gamma^\ell$ if and only if $\tilde{v}^\ell \in \tilde{\mathcal{T}}_\gamma^\ell$. Therefore,

$$\inf_{\hat{v}^\ell \in \hat{\mathcal{T}}_\gamma^\ell} \min_{i \in \mathbb{N}} d(\hat{v}^\ell, \text{Im } \hat{\beta}_i) = \inf_{\tilde{v}^\ell \in \tilde{\mathcal{T}}_\gamma^\ell} \min_{i \in \mathbb{N}} d(\tilde{v}^\ell, \text{Im } \tilde{\beta}_i) \quad (78)$$

Recalling that $\hat{\beta}_i = \hat{U}_\ell^* \hat{B}_i^{\ell-1}$ and $\tilde{\beta}_i = \tilde{U}_\ell^* \hat{B}_i^{\ell-1}$, we establish that (74) depends on \hat{U}_ℓ only through its orthogonal complement $\hat{v}_1^{\ell-1}$. Following similar considerations, we can establish that $\eta_\gamma^{k,\ell}$ depends on \hat{U}_r only through its orthogonal complement \hat{v}_1^{r-1} , for $r = k, \dots, \ell$.

To prove part b) of Claim 4, consider (74) again. Since $\eta_\gamma^{\ell,\ell} = \epsilon_d^{\ell,*}$ as in (29), we already know that $\eta_\gamma^{\ell,\ell} > 0$ for every possible $\hat{v}_1^{\ell-1}$. Next, for fixed $\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}$, we may write

$$\eta_\gamma^{\ell,\ell} = -g(\hat{v}_1^{\ell-1}) = \inf_{\hat{v}_1^{\ell-1} \in \mathcal{R}(\hat{U}_\ell)} -f(\hat{U}_\ell, \hat{v}_1^{\ell-1}) = - \sup_{\hat{v}_1^{\ell-1} \in \mathcal{R}(\hat{U}_\ell)} f(\hat{U}_\ell, \hat{v}_1^{\ell-1}), \quad (79)$$

$$\eta_\gamma^{\ell-1,\ell} = \inf_{\hat{v}_1^{\ell-1} \in \hat{\mathcal{T}}_\gamma^{\ell-1}} -g(\hat{v}_1^{\ell-1}), \quad (80)$$

where, in (79), \hat{U}_ℓ must be taken so that $[\hat{v}_1^{\ell-1} | \hat{U}_\ell]$ is unitary, and we have defined

$$f(\hat{U}_\ell, \hat{v}_1^\ell) = - \min_{i \in \underline{n}} d(\hat{v}_1^\ell, \text{Im } \hat{U}_\ell^* \hat{B}_i^{\ell-1}), \quad (81)$$

$$\mathcal{R}(\hat{U}_\ell) = \hat{\mathcal{T}}_\gamma^\ell(\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}, \hat{U}_\ell). \quad (82)$$

Note that f is continuous at every $(\hat{U}_\ell, \hat{v}_1^\ell)$ for which $\hat{U}_\ell^* \hat{B}_i^{\ell-1} \neq 0$ for $i \in \underline{n}$.

Let \mathcal{U} denote the following set

$$\mathcal{U} := \{\hat{U}_\ell \in \mathbb{C}^{n-\ell, n-\ell-1} : [\hat{v}_1^{\ell-1} | \hat{U}_\ell] \text{ is unitary, } \hat{v}_1^{\ell-1} \in \hat{\mathcal{T}}_\gamma^{\ell-1}\} \quad (83)$$

and regard \mathcal{R} as the set-valued map $\mathcal{R} : \mathcal{U} \rightsquigarrow \mathcal{S}_1$ that maps each $\hat{U}_\ell \in \mathcal{U}$ to the set $\mathcal{R}(\hat{U}_\ell) \subset \mathcal{S}_1$. Making these considerations, the function $g : \hat{\mathcal{T}}_\gamma^{\ell-1} \rightarrow \mathbb{R}$ defined above can be regarded as a marginal function. Application of Theorem 1.4.16 of [15] to g yields that g is upper semicontinuous if f and \mathcal{R} are upper semicontinuous, and \mathcal{R} has compact values. Since f is continuous, it is upper semicontinuous. \mathcal{R} takes compact values because $\hat{\mathcal{T}}_\gamma^\ell$ is compact.

Claim 5. *The set-valued map $\mathcal{R} : \mathcal{U} \rightsquigarrow \mathcal{S}_1$ is upper semicontinuous.*

Proof. Since \mathcal{R} has compact values, i.e. the sets $\mathcal{R}(\hat{U}_\ell)$ are compact for every $\hat{U}_\ell \in \mathcal{U}$, we need only prove that the graph of \mathcal{R} is closed. For, take a sequence $\{u_k \in \mathcal{U}\}_{k=0}^\infty$ so that $\lim_{k \rightarrow \infty} u_k = \hat{U}_\ell$ and suppose that $v_k \in \mathcal{R}(u_k)$ so that $\lim_{k \rightarrow \infty} v_k = v \in \mathcal{S}_1$. We have to establish that $v \in \mathcal{R}(\hat{U}_\ell)$. Recalling (42)–(43), we have that $v_k \in \mathcal{R}(u_k)$ implies

$$\hat{P}_{i,k}^\ell (\lambda_{i,k} I - u_k^* \hat{A}_i^{\ell-1} u_k) v_k = 0, \quad (84)$$

with $|\lambda_{i,k}| \leq 1 - \gamma$ and

$$\hat{P}_{i,k}^\ell = \left[I - u_k^* \hat{B}_i^{\ell-1} \left((u_k^* \hat{B}_i^{\ell-1})^* u_k^* \hat{B}_i^{\ell-1} \right)^{-1} (u_k^* \hat{B}_i^{\ell-1})^* \right]. \quad (85)$$

for all $i \in \underline{n}$ and $k \geq 0$. We may rewrite (84) as

$$\hat{P}_{i,k}^\ell v_k \lambda_{i,k} = \hat{P}_{i,k}^\ell u_k^* \hat{A}_i^{\ell-1} u_k v_k. \quad (86)$$

Since u_k and v_k are convergent, the limit of the right-hand side of (86) exists and we have

$$\lim_{k \rightarrow \infty} \hat{P}_{i,k}^\ell v_k \lambda_{i,k} = \hat{P}_i^\ell \hat{U}_\ell^* \hat{A}_i^{\ell-1} \hat{U}_\ell v = \lim_{k \rightarrow \infty} \hat{P}_i^\ell v \lambda_{i,k}, \quad (87)$$

where the last equality above follows because $\lambda_{i,k}$ is bounded. If $\hat{P}_i^\ell v \neq 0$, then $\lim_{k \rightarrow \infty} \lambda_{i,k} = \lambda_i$, with $|\lambda_i| \leq 1 - \gamma$. If $\hat{P}_i^\ell v = 0$, then (87) equals zero. In either case, we have $v \in \mathcal{R}(\hat{U}_\ell) = \hat{\mathcal{T}}_\gamma^\ell(\hat{A}_i^{\ell-1}, \hat{B}_i^{\ell-1}, \hat{U}_\ell)$. \square

The continuity of f and Claim 5 imply that g is upper semicontinuous. Then, $-g$ is lower semicontinuous and since $\hat{\mathcal{T}}_\gamma^{\ell-1}$ is compact, then $-g$ will attain a minimum within $\hat{\mathcal{T}}_\gamma^{\ell-1}$. Therefore, $\eta_\gamma^{\ell-1, \ell} > 0$. Following similar considerations, we can establish that $\eta_\gamma^{k, \ell} > 0$ for $k = 1, \dots, \ell$.

ii) By Definition 2, a SLASF $\hat{\mathcal{Z}}$ with compatible \hat{K}_i means that \hat{A}_i^{cl} will generate a solvable Lie algebra and satisfy $\rho(\hat{A}_i^{\text{cl}}) < 1$. In addition, part i) ensures that $\rho(\hat{A}_i^{\text{cl}}) \leq 1 - \epsilon_c/2$ irrespective of which ϵ or δ are selected. By Lemma 1, the closed-loop DTSS with subsystem matrices \hat{A}_i^{cl} admits a CQLF. Part ii) then follows from robustness of the CQLF, selecting $\epsilon > 0$ small enough.

References

- [1] D. Liberzon and A. S. Morse, “Basic problems in stability and design of switched systems,” *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.
- [2] A. P. Molchanov and Y. S. Pyatnitskiy, “Criteria of asymptotic stability of differential and difference inclusions encountered in control theory,” *Systems and Control Letters*, vol. 13, no. 1, pp. 59–64, 1989.
- [3] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, “Stability criteria for switched and hybrid systems,” *SIAM Review*, vol. 49, no. 4, pp. 545–592, 2007.
- [4] J. Daafouz, P. Riedinger, and C. lung, “Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach,” *IEEE Transactions on Automatic Control*, vol. 47, no. 11, pp. 1883–1887, 2002.
- [5] A. Sala, “Computer control under time-varying sampling period: an LMI gridding approach,” *Automatica*, vol. 41, no. 12, pp. 2077–2082, 2005.
- [6] K. Wulff, F. Wirth, and R. Shorten, “A control design method for a class of switched linear systems,” *Automatica*, vol. 45, no. 11, pp. 2592–2596, 2009.
- [7] H. Lin and P. J. Antsaklis, “Stability and stabilisability of switched linear systems: a survey of recent results,” *IEEE Trans. on Automatic Control*, vol. 54, no. 2, pp. 308–322, February 2009.
- [8] P. Kokotović and M. Arcak, “Constructive nonlinear control: a historical perspective,” *Automatica*, vol. 37, pp. 637–662, 2001.
- [9] H. Haimovich, J. H. Braslavsky, and F. Felicioni, “On feedback stabilisation of switched discrete-time systems via Lie-algebraic techniques,” in *Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, China, December 2009, pp. 1118–1123, dOI: 10.1109/CDC.2009.5399527. Also submitted to the *IEEE Trans. on Automatic Control*.
- [10] D. Liberzon, *Switching in systems and control*. Birkhäuser, 2003.
- [11] J. Theys, “Joint spectral radius: theory and approximations,” Ph.D. dissertation, Center for Systems Engineering and Applied Mechanics, Université catholique de Louvain, 2005.
- [12] K. Erdmann and M. Wildon, *Introduction to Lie algebras*. Springer-Verlag London, 2006.
- [13] Y. Mori, T. Mori, and Y. Kuroe, “A solution to the common Lyapunov function problem for continuous-time systems,” in *Proc. 36th IEEE Conf. on Decision and Control, San Diego, CA, USA*, 1997, pp. 3530–3531.
- [14] W. M. Wonham, *Linear multivariable control: a geometric approach*, 3rd ed. New York: Springer-Verlag, 1985.
- [15] J.-P. Aubin and H. Frankowska, *Set-valued analysis*. Birkhuser Boston, 1990.